PROPAGATION OF SMALL DISTURBANCES IN VAPOR-AND-LIQUID BUBBLING MEDIA

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We consider the problem of the propagation of small sinusoidal waves in a liquid containing vapor bubbles. From an analysis of the dispersion relation, we show the region of values of the parameters of the two-phase medium, as well as the disturbance frequencies at which the velocity of propagation of the disturbances is described by Landau's formula [1]. We show in this study that the equilibrium state of the two-phase bubbling vapor-and-liquid medium will be stable only if a certain condition imposed on the volumetric content and the dimensions of the bubbles is satisfied.

We analyze the influence of the heat exchange between the phases, the surface tension, the volumetric content and dimensions of the bubbles, and the frequency of the disturbances on the velocity of propagation and the damping coefficient. The asymptotic formulas obtained in this study enable us to estimate the velocity and the damping on the basis of simple analytic expressions.

For unstable mixtures, on the basis of our numerical calculations and the analytic formulas obtained, we investigate the influence of various parameters of the bubbling vapor-and-liquid medium on the coefficient determining how fast the amplitude of the disturbances increases.

It should also be noted that one purpose of the study is to refine the results obtained in [2], in which the heat exchange was taken into account by means of the effective Nusselt numbers.

The problem of the propagation of disturbances in a bubbling vapor-and-liquid medium, disregarding capillary phenomena, was considered in [3, 4]. The results of the latter study and the present one, with regard to the propagation of small disturbances, agree within the region of their common limits of applicability.

1. Fundamental Equations. Suppose that we have a mixture of a liquid with spherical vapor bubbles. In order to take account of the heat exchange between the phases, we shall use the heat-conduction equation written within the framework of spherical symmetry within and around the bubbles, as well as the system of boundary conditions for this equation, taking account of the phase transitions. Therefore, in addition to the usual macroscopic parameters introduced in the mechanics of multiphase media [5], we shall introduce microparameters characterizing the density and temperature distributions within and around the bubbles.

The system of macroscopic equations relating to the conservation of mass, of the number of bubbles, and of the momentum of the entire mixture for a plane single-velocity one-dimensional motion in the linear approximation has the form

$$\frac{\partial \rho_1}{\partial t} + \rho_{10}\frac{\partial v}{\partial x} = -I,\quad \frac{\partial \rho_2}{\partial t} + \rho_{20}\frac{\partial v}{\partial x} = I,\quad \frac{\partial n}{\partial t} + n_0\frac{\partial v}{\partial x} = 0,\quad \rho_0\frac{\partial v}{\partial t} + \frac{\partial p_1}{\partial x} = 0 \tag{1.1}$$

$$\left(I = 4\pi a_0^2 n_0 j,\quad \rho = \rho_1 + \rho_2,\quad \rho_i = \rho_i^0 \alpha_i,\quad \alpha_1 + \alpha_2 = 1,\quad \alpha_2 = \frac{4}{3}\pi a^3 n\right).$$

The subscripts i = 1, 2 refer, respectively, to the parameters of the liquid and the vapor; ρ_i, ρ_i^0, v, p, n, and a are, respectively, the disturbances in the density averaged over the mixture and over the phase, the velocity, the pressure, the number of bubbles per unit mixture volume, and the bubble radius; I and j are the intensities of mass exchange between the phases, referred to a unit volume of the mixture and to a unit area of the interface between the phases. The parameters corresponding to the undisturbed state have an additional subscript 0.

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We write the equations for the distribution of temperatures within and around the bubbles situated in a macroparticle with coordinate x:

$$\rho_{10}^0 c_1 \frac{\partial T_1'}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\lambda_1 r^2 \frac{\partial T_1'}{\partial r}\right) \quad (r > a_0),$$
$$\rho_{20}^0 c_{2p} \frac{\partial T_2'}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\lambda_2 r^2 \frac{\partial T_2'}{\partial r}\right) + \frac{\partial p_2}{\partial t} \quad (r < a_0), \tag{1.2}$$

where r is the microcoordinate, for which we take the distance from the center of the bubble; T_i' and λ_i are the temperatures and thermal conductivities of the phases; c_1, c_{2p}, and c_{2V} are, respectively, the specific heat of the liquid and those of the vapor at constant pressure and at constant volume. Here and hereafter, the primes indicate microparameters. It should be noted that the heat-conduction equation has been written without taking account of the compressibility of the liquid.

The equations of state of the phases will be taken in the form

$$p_1 = p_{10} + d_1^2(\rho_1^0 - \rho_{10}^0), \quad p_2 = \rho_2^{0\prime} B T_2', \tag{1.3}$$

where d_1 is the velocity of sound in the liquid (the equation of state of the liquid is written in the acoustic approximation); B is the gas constant.

The equation of pulsating motion, disregarding the crowding of the bubbles, has the form

$$a\frac{\partial w_{1a}}{\partial t} + 4\nu_1 w_{1a}/a_0 = \left(p_2 - p_1 + \frac{2\sigma}{a_0}\frac{a}{a_0}\right)\Big/\rho_{10}^0, \tag{1.4}$$

where the w_{ia} are the velocities of pulsating motion of the phases on the interface; σ is the coefficient of surface tension; ν_1 is the kinematic viscosity of the liquid.

On the interface between the phases we specify the following boundary conditions:

$$T_1' = T_2' = T_a = T_s(p_2), \quad \lambda_1 \frac{\partial T_1'}{\partial r} - \lambda_2 \frac{\partial T_2'}{\partial r} = jl,$$
$$\rho_{20}^0\left(\frac{\partial a}{\partial t} - w_{2a}\right) = \rho_{10}^0\left(\frac{\partial a}{\partial t} - w_{1a}\right) = j \quad (r = a_0), \tag{1.5}$$

where l is the specific heat of vaporization; $T_s(p_2)$ is the saturation temperature at pressure p_2.

Furthermore,

$$\partial T_2'/\partial r = 0 \quad (r = 0). \tag{1.6}$$

To close the system of boundary conditions for the heat-conduction equation, we must specify one more condition for T_1'. In those cases in which the temperature drops in the liquid close to the boundary between the phases take place at distances much smaller than the average distance between bubbles, we can set

$$T_1' = T_0 \quad (r = \infty).$$

This condition means that the liquid far from the bubbles is taken as a thermostat (the condition of isothermality far from the bubbles).

We consider the case in which the lengths of the temperature waves initiated by the radial motions of the bubbles are comparable to the average distances between bubbles. We assume that there is no return flow of heat between masses of liquid associated with adjacent inclusions (i.e., that all the bubbles for a fixed macroparticle are of equal rank). This condition, within the framework of spherical symmetry, can be written as

$$\partial T_1'/\partial r = 0 \quad (r = a_*). \tag{1.7}$$

As our a_* we take the radius of a spherical cell [5], expressed by the formula

$$a_* = a_0/\alpha_{20}^{1/3}. \tag{1.8}$$

According to the boundary condition adopted above, we assume that the mass of liquid associated with one bubble is situated in a sphere of radius a_* and that on this sphere the heat flux is equal to zero (the condition of adiabaticity of a cell).

For the values of the parameters on the interface between the phases, we write the Claperon–Clausius equation

$$\frac{dp_a}{dT_a} = \frac{l}{T_a\left(1/\rho_{2a}^0 - 1/\rho_{1a}^0\right)} \quad (p_a = p_2). \tag{1.9}$$

In solving the problem it is convenient to use the relation for p_2 that was obtained from the equation of heat conduction by assuming homobaricity [5]:

$$\frac{\partial p_2}{\partial t} = -\frac{3\gamma}{a_0}\left[p_{20}w_{2a} - (1 - 1/\gamma)\lambda_2\left(\frac{\partial T_2'}{\partial r}\right)_a\right] \quad (\gamma = c_{2p}/c_{2V}). \tag{1.10}$$

2. Solution of the Problem. On the basis of the above system (1.1)-(1.10), we consider the problem of the propagation of small disturbances. We shall seek the solution in the form of a damped traveling wave:

$$p, v, w, a, n \sim \exp\left[i(Kx - \omega t)\right],\ T' = T^A(r)\exp\left[i(Kx - \omega t)\right], \tag{2.1}$$
$$K = k + i\delta,\ d_p = \omega/k,$$

K is the wave vector (a complex number, where δ and d_p are, respectively, the damping coefficient and the phase velocity of the wave, which are determined by the imaginary and real parts of the wave vector).

From the condition that a solution of this form exists, we obtain the dispersion equation

$$\frac{K^2}{\omega^2} = \rho_0\left(\alpha_{10}/\rho_{10}^0 d_1^2 + 3\alpha_{20}/\psi\right),\quad \psi = 3/\Pi - \rho_{10}^0\omega^2 a_0^2 - 4i\rho_{10}^0\nu_1\omega - 2\sigma/a_0, \tag{2.2}$$

$$\Pi = \frac{1 + i\varkappa_2\left[\Pi_2 Q\left(1 - \chi(1 - s)\right) + \eta(1 - s)(1 + \Pi_1 Y)\right]/\omega a_0^2}{\gamma p_{20} + i\varkappa_2 s(2\sigma/3a_0)\left[\Pi_2 Q\chi - \eta(1 + \Pi_1 Y)\right]/\omega a_0^2},$$

$$\chi = c_{2p}T_0/l,\quad Y = \left(-i\omega a_0^2/\varkappa_1\right)^{1/2},\quad Z = \left(-i\omega a_0^2/\varkappa_2\right)^{1/2},$$

$$\Pi_1 = \frac{AY\,\mathrm{th}\left[Y(A - 1)\right] - 1}{AY - \mathrm{th}\left[Y(A - 1)\right]},\quad \Pi_2 = Z\,\mathrm{cth}\,Z - 1,\quad \varkappa_1 = \lambda_1/\rho_{10}^0 c_1,$$

$$\varkappa_2 = \lambda_2/c_{2p}\rho_{20}^0,\quad A = 1/\alpha_{20}^{1/3},\quad p_{20} = p_{10} + 2\sigma/a_0,$$

$$s = \rho_{20}^0/\rho_{10}^0,\quad T_0 = T_s(p_{20}),\quad Q = 3(1 - \chi)(\gamma - 1),\quad \eta = 3(\gamma - 1)\chi^2\lambda_1/\lambda_2.$$

For the distribution of the temperatures, the pressure in the bubbles, the bubble radius, and the velocity, we have the relations

$$\frac{T_1^A}{T_0} = \chi(1 - 1/\gamma)\frac{\left[(AY + 1)e^{Y(R - A)} + (AY - 1)e^{-Y(R - A)}\right]}{R\left[(AY + 1)e^{Y(1 - A)} + (AY - 1)e^{-Y(1 - A)}\right]}\frac{p_2}{p_{20}},$$
$$R = r/a_0\ (1 < R < A),$$

$$\frac{T_2^A}{T_0} = (1 - 1/\gamma)\left[\frac{(\chi - 1)\,\mathrm{sh}(ZR)}{R\,\mathrm{sh}(Z)} + 1\right]\frac{p_2}{p_{20}}\quad (R < 1),$$

$$\frac{p_2}{p_{20}} = -\left\{3\gamma/\left[1 + i\varkappa_2\left(\Pi_2 Q(1 - \chi) + \eta(1 + \Pi_1 Y)\right)/\omega a_0^2\right]\right\}\frac{a}{a_0},$$

$$\frac{a}{a_0} = -\frac{p_1}{\psi_1},\quad \psi_1 = \psi\left\{1 + s\frac{i\varkappa_2\left[Q\Pi_2\chi - \eta(1 + \Pi_1 Y)\right]/\omega a_0^2}{1 + i\varkappa_2\left[\Pi_2 Q(1 - \chi) + \eta(1 + \Pi_1 Y)\right]/\omega a_0^2}\right\},\quad v = \frac{K}{\omega}\frac{p_1}{\rho_0}.$$

On the basis of (2.2), passing to the limit as $\omega \to 0$, we obtain for the equilibrium velocity of sound the expression

$$d_e = \left[1/d_1^2 + 1/d_L^2\left(1 - \alpha_{10}D/\alpha_{20}a_0\right)\right]^{-1/2}/\alpha_{10},$$

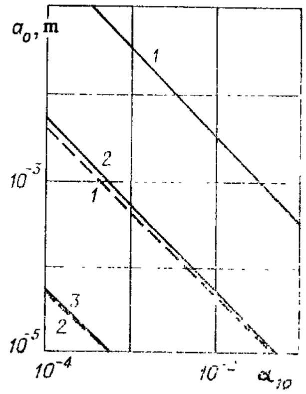

Fig. 1

$$d_L^2 = \frac{\gamma p_{20}}{\rho_{10}^0}\left[1 + \frac{2\sigma}{3a_0 p_{20}} s(1 - 1/\gamma)\left(\chi(1-\chi) - \frac{\rho_{10}^0 c_1}{\rho_{20}^0 c_{2p}}\chi^2\right)\right]\left\{\frac{\alpha_{20}}{\alpha_{10}} + (\gamma - 1)\times\right.$$
$$\left.\times\left[(1-\chi)(1-\chi(1-s)) + \frac{\rho_{10}^0 c_1}{\rho_{20}^0 c_{2p}}\chi^2(1-s)\right]\right\}^{-1}, \quad D = 2\sigma/3\rho_{10}^0 d_L^2,$$

which is valid if the parameters of the medium satisfy the relation

$$1/d_1^2 + 1/d_L^2(1 - \alpha_{10}D/\alpha_{20}a_0) \geqslant 0. \tag{2.3}$$

In the case when (2.3) is not satisfied, $d_e = \infty$. Since we usually have

$$\chi \sim 1, \quad \frac{\rho_{10}^0 c_1}{\rho_{20}^0 c_{2p}} \gg \frac{\alpha_{20}}{\alpha_{10}},$$

it follows that the expressions for d_L and D can be simplified:

$$d_L^2 = \frac{p_{20}}{(1 - 1/\gamma)\rho_{10}^0}\frac{\rho_{20}^0 c_{2p}}{\rho_{10}^0 c_1}\left(\frac{l}{c_{2p}T_0}\right)^2, \quad D = \frac{2}{3}\Sigma\beta, \quad \Sigma = \frac{\sigma}{\gamma p_{20}}, \tag{2.4}$$
$$\beta = 3(\gamma \quad 1)\frac{\rho_{10}^0 c_1}{\rho_{20}^0 c_{2p}}\chi^2.$$

If we set $p_{20} \approx p_{10}$, then the expression for d_L coincides with the well-known formula obtained by Landau [1] for the velocity of sound in a vapor-and-liquid medium for low values of mass vapor content. Thus, the equilibrium velocity of sound in the liquid with vapor bubbles can be described by Landau's formula only when we can disregard the surface tension and the compressibility of the liquid.

The velocity value obtained by formula (2.4) is small if the state of the vapor-and-liquid medium is not close to critical. For a steam-and-water mixture, for example, when $p_{20} = 10^5$ Pa, we have $d_L = 1.1$ m/sec ($D = 0.35\cdot 10^{-4}$ m). Therefore in the expression for the equilibrium velocity, we can disregard the compressibility of the liquid, and for a fairly wide range of values of the parameters of the two-phase medium we can set

$$d_e = \frac{d_L}{\alpha_{10}}(1 - \alpha_{10}D/\alpha_{20}a_0)^{1/2} \quad (\alpha_{20}a_0/\alpha_{10} \geqslant D), \quad d_e = \infty \quad (\alpha_{20}a_0/\alpha_{10} < D).$$

For the above-mentioned values of d_L and D, when $\alpha_{20} \lesssim 10^{-2}$ (which is characteristic of bubbling media) and $a_0 \lesssim 10^{-3}$ m, the equilibrium velocity is infinite.

3. Analysis of the Dispersion Relation. On the basis of the dispersion expression (2.2), we can analyze the stability of the equilibrium state of the medium with respect to small sinusoidal disturbances. The relation (2.2) must be regarded as an equation in ω for real values of K. Using the argument principle, as it was used in [6], we can show that this equation for ω has an imaginary root $\omega = \omega' i$ ($\omega' > 0$), if

$$\alpha_{20}a_0/\alpha_{10} < D. \tag{3.1}$$

Since in obtaining (3.1) we are looking for a solution in the form (2.1), the existence of such a root with a positive imaginary part means that the amplitudes of the disturbances increase beyond all bound as time increases. Therefore the equilibrium two-phase state is unstable if the condition (3.1) is satisfied.

We give below the results of the numerical calculation, as well as analytic expressions for the variation of ω', which characterizes the growth of the amplitude of the imposed disturbances as time increases ($t' = 1/\omega'$ is the time required for the amplitude to increase by a factor of e), as a function of the wave number K and of the parameters of the medium.

Thus, for the existence of a finite velocity, and also for stability of the equilibrium state of the bubbling vapor-and-liquid medium, the values of the volumetric content and radius of the bubbles must satisfy the inequality $\alpha_{20}a_0/\alpha_{10} \geqslant D$.

The parameter D, in general, depends on the bubble radius, since $T_0 = T_S(p_{20})$, $p_{20} = p_{10} + 2\sigma/a_0$. However, the variation of temperature as the radius varies from 10^{-3} to 10^{-6} m, for most substances, is several degrees. Therefore we may assume that D depends only on the pressure p_{10}. In Fig. 1 the regions lying above the straight lines correspond to the values $p_{10} = 10^5$, 10^6, and 10^7 Pa for a steam-and-water mixture (solid lines) and for a bubbling vapor-and-liquid mixture in nitrogen (dashed lines). As can be seen, for identical pressures p_{10} the region of values of the parameters α_{20}, a_0 for which there exists a finite equilibrium velocity of sound (and the two-phase state is stable) in the case of nitrogen is much broader than the region for water.

If we want the equilibrium velocity to be described by Landau's formula, the condition $\alpha_{20}a_0/\alpha_{10} \gg D$ must be satisfied, i.e., the values of the parameters α_{20} and a_0 must lie far deeper inside the stability region. This can be ensured by passing to higher pressures, and also by choosing sufficiently large values of α_{20} and a_0.

It should be noted that if the condition of adiabaticity of a cell is replaced by the condition of isothermality and we take account of capillary effects, then the equilibrium velocity will be infinite, and therefore in such a scheme the bubbling vapor-and-liquid mixture will always be unstable.

We shall give an estimate for the frequencies at which velocity values close to the equilibrium value will be realized. An analysis of the dispersion relation (2.2) shows that for this, the condition

$$|Z| \ll 1, \quad |Y|(A-1) \ll 1 \tag{3.2}$$

must be satisfied. For most media the second condition is stronger, and this means that the lengths of the temperature waves initiated by the radial motions of bubbles are comparable with the distances between the bubbles. From (3.2) we have

$$\omega^{1/2} \ll \omega_*^{1/2} = \omega_T^{1/2}/(1/\alpha_{20}^{1/3} - 1), \quad t_* = 2\pi/\omega_* \quad (\omega_T = \varkappa_1/a_0^2), \tag{3.3}$$

t_* is the characteristic time required for the temperature waves around the bubbles to traverse distances comparable to the distances between bubbles. In the case of water, for example, when $a_0 = 10^{-5}$ m, $\alpha_{20} = 10^{-2}$, we have $\omega_* \approx 10^2$ sec^{-1}, $t_* = 0.05$ sec ($\varkappa_1 = 1.6 \cdot 10^{-7}$ m^2/sec).

We shall also give the asymptotic expressions for the phase velocity and the damping coefficient when the condition (3.3) is satisfied. The expression for Π can be simplified if we consider that for most substances, over a wide range of the parameters being varied, we have the estimates

$$0 < Q < 1, \ 0 < \chi < 1, \ \eta \approx 1, \ s \ll 1, \ |\Pi_2| \ll \eta|1 + \Pi_1 Y|.$$

Physically the meaning of the last inequality is that when there are phase transitions, the internal thermal problem becomes unimportant. After simplifying, we find

$$\Pi = [Y^2 + \beta(1 + \Pi_1 Y)]/\gamma p_{20} Y^2.$$

As can be seen from the results of investigations on the dynamics of steam bubbles as well as from estimates and calculations based on the dispersion relation (2.2), viscosity effects are usually small, and in what follows, we shall disregard them. It should be noted that if (3.2) is satisfied, we must necessarily have

$$\rho_{10}^0\omega^2 a_0^2 \ll 2\sigma/a_0.$$

Then, taking account of the aforementioned simplifications, the dispersion relation will take the form

$$\frac{\omega^2}{K^2} = \frac{d_L^2}{\alpha_{10}^2}(1-\alpha_{10}D/\alpha_{20}a_0 - iM\omega/\omega_T), \quad M = \frac{(A-1)^3(5A^3+6A^2+3A+1)}{15(A^3-1)}.$$

Hence, for frequencies satisfying the relation

$$\omega/\omega_T \ll |1-\alpha_{10}D/\alpha_{20}a_0|/M,$$

we have

$$d_p = \frac{d_L}{\alpha_{10}}(1-\alpha_{10}D/\alpha_{20}a_0)^{1/2}(1+O(\omega^2)), \quad \delta = \frac{1}{2}\frac{\alpha_{10}M}{d_L(1-\alpha_{10}D/\alpha_{20}a_0)^{3/2}}\frac{\omega^2}{\omega_T^2} \tag{3.4}$$

$$(\alpha_{10}a_0/\alpha_{20} > D),$$

$$d_p = \frac{2d_L(\alpha_{10}D/\alpha_{20}a_0-1)^{1/2}}{\alpha_{10}M}\frac{\omega_T}{\omega}, \quad \delta = \frac{\alpha_{10}\omega}{d_L(\alpha_{10}D/\alpha_{20}a_0-1)^{1/2}}$$

$$(\alpha_{10}a_0/\alpha_{20} < D).$$

According to the second group of formulas in (3.4), as the frequency decreases, the phase velocity will increase beyond all bounds, while the damping coefficient will tend to zero, i.e., the low-frequency components of the disturbances will be propagated at higher velocities when the damping is low, which is also an indication of instability of the medium.

Let us consider the range of frequencies satisfying the condition $|Y|(A-1) \gg 1$ for which the temperature drops will take place at distances much smaller than the average distance between bubbles; then $\Pi_1 \approx 1$, and the expression for Π takes the form

$$\Pi = [Y^2 + \beta(1+Y)]/\gamma p_{20}Y^2.$$

Suppose, furthermore, that $|Y| \gg 1$, $|Y^2| \gg \beta|Y|$. The first condition means that the temperature drops in the liquid takes place in thin layers near the interface between the phases. The last condition is stronger, and if it is satisfied, i.e., if

$$\omega^{1/2} \gg \omega_{**}^{1/2} = \beta\omega_T^{1/2}, \tag{3.5}$$

the dispersion expression will have the form

$$\frac{K^2}{\omega^2} = \alpha_{10}\{\alpha_{10}/d_1^2 + 3\rho_{10}^0\alpha_{20}[3\gamma p_{20}(1-\beta/Y) - \rho_{10}^0\omega^2 a_0^2 - 2\sigma/a_0]^{-1}\}. \tag{3.6}$$

Then for frequencies much lower than the resonance Minnaert frequency [6], i.e., when

$$\omega^2 \ll \omega_0^2 = 3\gamma p_{20}/\rho_{10}^0 a_0^2, \tag{3.7}$$

for the phase velocity and the samping coefficient, disregarding capillary effects, we have

$$d_p = [\alpha_{10}^2/d_1^2 + \rho_{10}^0\alpha_{10}\alpha_{20}/\gamma p_0]^{-1/2}, \tag{3.8}$$
$$\delta = (1/2)^{3/2}\alpha_{20}\beta d_p(\rho_{10}^0/p_0)(\omega\omega_T)^{1/2} \quad (p_{10} = p_{20} = p_0).$$

If the volumetric content of the bubbles is not very small $(\alpha_{20} \gg \alpha_{2*} = \gamma p_0/\rho_{10}^0 d_1^2)$, then

$$d_p = d_T = (\gamma p_0/\rho_{10}^0\alpha_{20}\alpha_{10})^{1/2}. \tag{3.9}$$

It should be noted that the above approximations for the phase velocity (3.8) and (3.9), correspond to the condition of zero heat and mass exchange between the phases and coincide with the expressions for the velocity of sound in the liquid with vapor bubbles for the case of adiabatic behavior. When $p_0 = 10^5$ Pa and $a_0 = 10^{-3}$ m, the estimates (3.5) and (3.7), for which formulas (3.8) and (3.9) are valid, yield 10^{-3} sec$^{-1} \lesssim \omega \lesssim 10^4$ sec^{-1} for water. As the pressure p_0 increases, and also as we pass to larger bubbles, this range will become broader.

For a phase velocity and damping coefficient close to the resonancefrequency ($\omega \approx \omega_0$), also disregarding the compressibility of the liquid, we obtain

$$d_p = d_T[2\varepsilon/(\sqrt{2}-1)]^{1/2}, \quad \delta = \omega_0/d_T[2\varepsilon(\sqrt{2}-1)]^{1/2} \qquad (3.10)$$
$$(\varepsilon = \beta(\omega_T/\omega_0)^{1/2}).$$

For a broad range of values of the parameters a_0 and p_0, we usually have $\varepsilon \ll 1$.

The next characteristic frequency beyond the resonance frequency (ω_0) is the frequency at which the phase velocity has a maximum (usually an anomalously high one). Analysis of formula (3.6) shows that the value of this frequency can be determined from

$$\omega_{00}^2 = \omega_0^2(1+\xi) \quad (\xi = \alpha_{20}/\alpha_{2*}\alpha_{10}). \qquad (3.11)$$

As the calculations show, in the range $\omega_0 \lesssim \omega \lesssim \omega_{00}$ there is an anomalously high damping of the disturbances, i.e., this range is a band of opacity. From (3.11) it follows that as the volumetric content of the bubbles increases, this region will become larger.

For frequencies greater than ω_{00} we can write the expression

$$d_p = \left[\alpha_{10}^2/d_1^2 + 3\alpha_{20}\alpha_{10}/a_0^2(\omega_0^2-\omega^2)\right]^{-1/2}, \qquad (3.12)$$
$$\delta = \frac{3\alpha_{20}d_1}{2\sqrt{2}\,a_0^2}\frac{\omega_0^2}{(\omega_0^2-\omega^2)^2}\beta(\omega\omega_T)^{1/2}.$$

It should be noted that in obtaining the asymptotic formulas (3.8)-(3.12), we assumed that the characteristic frequencies ω_{**} and ω_0 satisfy the relation $\omega_{**} \ll \omega_0$, from which it follows that the bubble radius is subject to the limitation

$$a_0 \gg a_* = \beta^2\varkappa_1(\rho_{10}^0/\beta\gamma p_0)^{1/2}. \qquad (3.13)$$

For a steam-and-water mixture, when $p_0 = 10^5$ Pa, for example, we find that $a_0 = 10^{-4}$ m. Consequently, the analytic formulas given above are valid for sufficiently large bubbles which satisfy the condition (3.13).

Thus, for vapor-and-liquid mixtures with large bubbles ($a_0 \gg a_*$), there are four characteristic frequencies. For frequencies $\omega \gg \omega_*$, the velocities of propagation of the disturbances can be described by formulas which are analogous for bubbling gas-and-liquid media in the case of "frozen" heat exchange. For small frequencies, when the temperature drops between the phases take place at distances comparable to the distances between bubbles ($\omega \ll \omega_*$) and when the values of the parameters a_0 and α_{20} correspond to the stable state of equilibrium, the phase velocity is close to the equilibrium velocity. In those cases when we can disregard the capillary effects, the equilibrium velocity is determined by Landau's formula. It should also be noted that for bubbling mixtures with $a_0 = 10^{-4}$–10^{-3} m, which are usually of the greatest practical interest, the equilibrium velocity is reached at very low frequencies.

If the values of the parameters a_0 and α_{20} in the region of unstable states lie sufficiently close to the boundary of this region, then as the frequency decreases from about the value $\omega = \omega_*$, the phase velocity will increase beyond all bounds. However, if the values of the parameters a_0 and α_{20} lie sufficiently deep inside the region of instability, then the phase velocity will begin to grow earlier, at some value $\omega = \omega_\sigma$, which is usually much larger than ω_*. For an estimate of ω_σ, we consider the dispersion expression in the frequency range

$$\omega_* \ll \omega \ll \omega_{**},$$

and in addition, let

$$\rho_{10}^0\omega^2 a_0^2 \ll 2\sigma/a_0.$$

Then the dispersion expression, if we disregard the compressibility of the liquid, can be written in the form

$$K^2/\omega^2 = 1/d_T^2(Y/\beta - \Sigma_*), \quad \Sigma_* = (2/3)\Sigma/a_0.$$

From an analysis of this expression, we find

$$\omega_\sigma = (2\beta\Sigma_*)^2\omega_T.$$

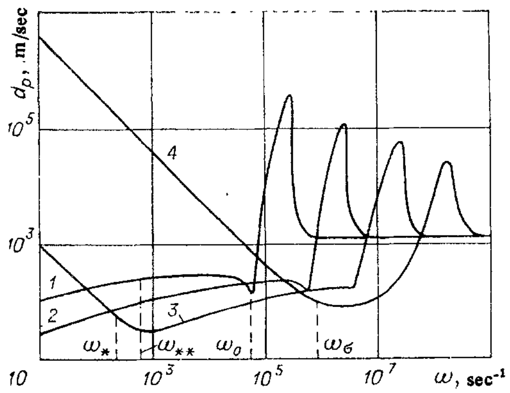

Fig. 2

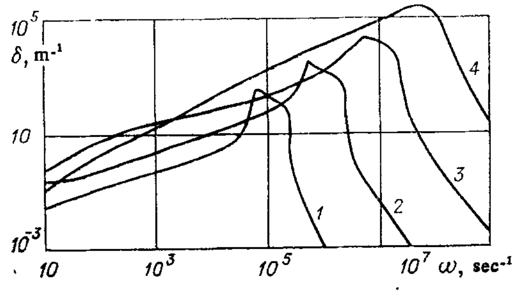

Fig. 3

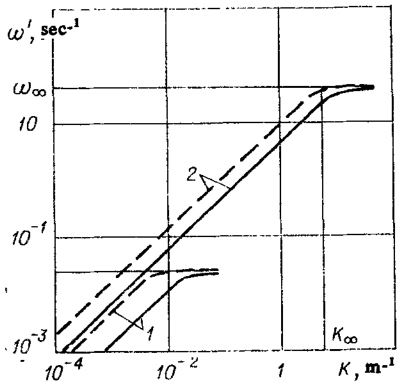

Fig. 4

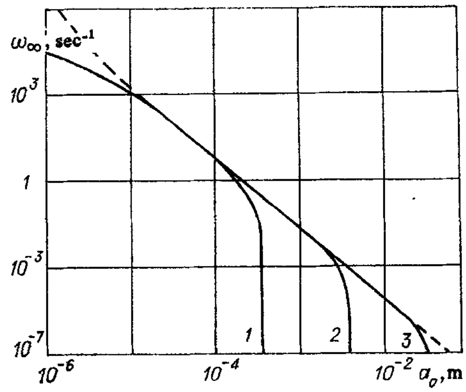

Fig. 5

The estimate so obtained is in good agreement with the numerical calculations. It should also be noted that the expression for ω_σ, to within a coefficient of 4, coincides with the frequency of the resonance caused by the capillary effects and the phase transitions for a unit bubble [6].

Figures 2 and 3 show the dispersion curves calculated for a bubbling steam-and-water medium when $p_{10} = 10^6$ Pa and $\alpha_{20} = 10^{-2}$. All the necessary thermophysical parameters have been estimated on the basis of [7]. Curves 1-4 correspond to bubble radii of $a_0 = 10^{-3}$, 10^{-4}, 10^{-5}, and 10^{-6} m. For radii of $a_0 = 10^{-3}$, 10^{-4} m the mixture is stable, whereas for $a_0 = 10^{-5}$, 10^{-6} M it is unstable. On the graphs for the phase velocity (see Fig. 2) we have marked the characteristic frequencies calculated by the formulas given above. It can be seen that the curves confirm those characteristic features for the phase velocity and the damping coefficient which were stated earlier on the basis of the asymptotic formulas we had obtained.

4. Investigation of Unstable States. As already noted above, formula (2.2), regarded as an equation for determining ω for parameters corresponding to the unstable state, has an imaginary solution with a positive imaginary part ($\omega = \omega' i$, $\omega' > 0$). Figure 4 shows the variation of ω' as a function of the wave number K for a bubbling steam-and-water mixture when $p_{10} = 10^5$ Pa. Curves 1 and 2 correspond to $a_0 = 10^{-3}$ and 10^{-4} m, the solid curves to $\alpha_{20} = 10^{-2}$, and the dashed curves to $\alpha_{20} = 10^{-3}$. An analysis of these curves shows that as the wave number K varies from zero to infinity, the parameter ω' will increase from zero to some maximum value ω_∞. If K is much smaller than some K_∞, which depends on the parameters of the two-phase medium, then for the relation between ω' and K, disregarding the compressibility of the liquid, we can write

$$\omega' = K d_L (\alpha_{10} D / \alpha_{20} a_0 - 1)^{1/2} / \alpha_{10}.$$

As our estimate for K_∞, we use the expression

$$K_\infty = \omega_\infty \alpha_{10} / d_L (\alpha_{10} D / \alpha_{20} a_0 - 1)^{1/2}.$$

If K increases further, beyond the value K_∞, there will be practically no further increase in ω'. As can be seen from the graphs, a decrease in the volumetric content, and also in the dimension of the bubbles, for fixed K, will lead to an increase in ω', the parameter determining how fast the amplitude of the disturbances increases.

In our analysis of the stability of the equilibrium, it is most important to determine the maximum ω_∞, which in this case is the root of the equation

$$\frac{3\gamma p_{20}Y^2}{Y^2+\beta(1+\Pi_1 Y)}+\rho_{10}^0\omega_\infty^2 a_0^2+4\nu_1\rho_{10}^0\omega_\infty-2\sigma/a_0=0,\quad Y=(\omega_\infty/\omega_T)^{1/2}. \tag{4.1}$$

The solution of this equation, satisfying conditions analogous to (3.2), has the form

$$\omega_\infty=(1-\alpha_{20}a_0/\alpha_{10}D)\,\omega_T/M.$$

This solution satisfies the condition for which it is obtained if $(1-\alpha_{20}a_0/\alpha_{10}D)\ll 1$, i.e., when the parameters α_{20} and a_0 lie sufficiently close to the boundary in the region of instability. When $p_{10}=10^5$ Pa, $a_0=10^{-3}$ m, and $\alpha_{20}=3\cdot 10^{-2}$, for example, we have $t_*=t'=$ 200 sec $(t'=1/\omega_\infty)$.

If α_{20} and a_0 lie sufficiently deep inside the region of instability, then, setting $\rho_{10}^0\omega_\infty^2 a_0^2\ll 2\sigma/a_0$, $\Pi_1\approx 1$, we find from Eq. (4.1) that

$$\omega_\infty=\frac{\Gamma^2\left[1+(1+4(1-\Sigma_*)/\Gamma)^{1/2}\right]^2\omega_T}{4(1-\Sigma_*)}\quad(\Gamma=\Sigma_*\beta). \tag{4.2}$$

For $p_{10}=10^5$ Pa, $a_0=10^{-3}$ and 10^{-4} m, respectively, we have $t'=50$ and 0.02 sec. Hence we can see that for sufficiently coarsely dispersed media ($a_0\gtrsim 10^{-3}$ m), the characteristic times t', from the practical point of view, are much larger, and in many cases such media can be regarded as stable. Figure 5 shows the variation of ω_∞ as a function of the bubble radius when $p_{10}=10^5$ Pa; curves 1-3 correspond to volumetric concentrations $\alpha_{20}=10^{-1}$, 10^{-2}, and 10^{-3}, and the dashed curve corresponds to formula (4.2). If the values of the parameters of the bubbling mixture are not very close to the boundary of instability, and if the bubbles are not very small, then the solution of Eq. (4.1) is determined fairly accurately by formula (4.2). For very small bubbles

$$a_0\ll a_{**}=\left(\rho_{10}^0\varkappa_1^2\Gamma_*^4/\sigma\right)^{1/5}\quad(\Gamma_*=(2/3)\,\Sigma\beta),$$

and for the root, we have

$$\omega_\infty=\left(2\sigma/\rho_{10}^0a_0^3\right)^{1/2}.$$

Thus, in bubbling vapor-and-liquid media, the effects resulting from the phase transitions and the capillary phenomena will lead to new theoretical features of the behavior of the bubbling mixtures. In particular, unlike bubbling gas-and-liquid media, a vapor-and-liquid bubbling mixture can be unstable. The tendency toward instability increases (i.e., ω_∞ becomes larger) as we pass to more finely dispersed mixtures and to mixtures with small volumetric content of the bubbles. Therefore any very finely dispersed bubbling vapor-and-liquid mixture will be strongly unstable and consequently difficult to 'contain.'

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